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HEAT CONDUCTION AND HEAT EXCHANGE IN TECHNOLOGICAL PROCESSES

TEMPERATURE FIELD NEAR THE SINGULAR LINE OF THE INTERFACE OF ANISOTROPIC MEDIA

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A representation of the temperature fields and the components of the vector of heat-flux density in both the base material (matrix) and the conjugate medium (inclusion) has been found for anisotropic media in the case of the interface with a singular line on condition of ideal thermal contact. It has been shown that the anisotropy of thermal properties makes it possible to do away with the singularity of the components of the heat-flux-density vector. Particular cases of isotropy of the media and of heat-insulated and isothermal inclusions have been investigated. The results obtained are applicable for studying the nonstationary heat conduction of an anisotropic body with an uneven anisotropic inclusion.

Keywords: *singular line, anisotropy of thermal properties, heat conduction, conjugation problem, singularity, distribution of the components of heat-flux density, temperature distribution, heat-flux coefficients.*

Introduction. The distribution of local temperature fields and of the components of the vector of heat-flux density near the singular line of the interface of anisotropic media makes it possible to construct temperature fields in anisotropic bodies with an uneven interface of the media, such as has been done for isotropic bodies [1, 2]. Therefore, knowledge of the asymptotics of the local temperature fields and the heat-flux density is necessary for constructing single- and double-layer potentials, which provides a mathematical basis for the methods of investigation of the temperature fields of structurally inhomogeneous anisotropic members.

Formulation of the Problem. Let the interface of anisotropic heat-conducting media be $S = S_1 \cup S_2$ (Fig. 1). The intersection of smooth surfaces S_1 and S_2 determines a smooth singular line L which is a set of angular points. We stretch a smooth surface S_0 over the line L; we introduce orthogonal curvilinear coordinates u, v on this surface so that the singular line is determined for $v = v_0 = \text{const}$. We assume [1] that parameterization results in $u = s$. We consider the moving trihedral $\mathbf{nn}_1\mathbf{n}_2$ of the interface S_0 at the point $M_0 \in S_0$. The point M of the vector plane $\mathbf{nn}_2$ will be determined by the polar radius ρ and by the angle $\theta = \angle\,(M_0M, \mathbf{n}_2)$. The variables ρ, θ, and s are curvilinear orthogonal coordinates determined by the relation

$$\mathbf{r} = \mathbf{r}_0 + \rho\,(\cos\theta\,\mathbf{n}_2(s) + \sin\theta\,\mathbf{n}(s))\,, \tag{1}$$

where the Lamé coefficients are $h_1 = 1$, $h_2 = \rho$, and $h_2 = 1 + \rho H_0$; $H_0 = -(p\cos\theta + r_1\sin\theta)$; $p = |\mathbf{r}_s|_v/|\mathbf{r}_v|$; $r_1 = (\mathbf{n}, \mathbf{r}_{ss}/|\mathbf{r}_s|$. The medium V_0 is characterized by the thermal conductivities $k_{11}^{(0)}$, $k_{22}^{(0)}$, and $k_{33}^{(0)}$, and the medium V_1 (inclusion) is characterized by the thermal conductivities $k_{11}^{(1)}$, $k_{22}^{(1)}$, and $k_{33}^{(1)}$.

The temperature fields are determined by the heat-conduction equations [3]

$$\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\rho}\left(\frac{k_{11}^{(i)}h_2h_3}{h_1}\frac{\partial T_i}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\frac{k_{22}^{(i)}h_1h_3}{h_2}\frac{\partial T_i}{\partial\theta}\right)+\frac{\partial}{\partial s}\left(\frac{k_{33}^{(i)}h_2h_1}{h_3}\frac{\partial T_i}{\partial s}\right)\right]=c_i\rho_{0i}\frac{\partial T_i}{\partial t}-Q_i\,. \tag{2}$$

The heat-conduction equation must hold at all points of the matrix, including those of the singular line

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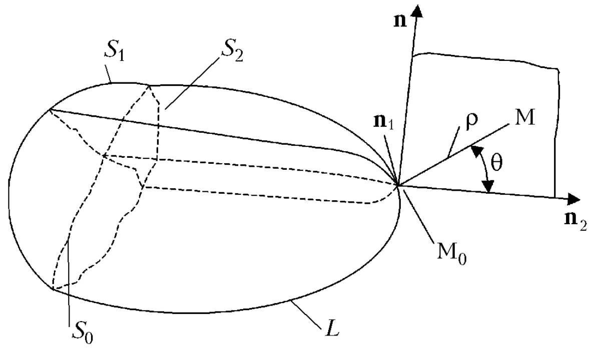

Fig. 1. Anisotropic media with the singular line of the interface of the media and the system of local curvilinear coordinates.

$$\lim \left\{ \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial \rho} \left(\frac{k_{11}^{(i)} h_2 h_3}{h_1} \frac{\partial T_i}{\partial \rho} \right) + \frac{\partial}{\partial \theta} \left(\frac{k_{22}^{(i)} h_1 h_3}{h_2} \frac{\partial T_i}{\partial \theta} \right) + \frac{\partial}{\partial s} \left(\frac{k_{33}^{(i)} h_2 h_1}{h_3} \frac{\partial T_i}{\partial s} \right) \right] - c_i \rho_{0i} \frac{\partial T_i}{\partial t} \right\} = 0 . \tag{3}$$

We assume that the conditions of ideal thermal contact [4]

$$T_0 - T_1 = 0 , \quad \mathbf{q}_{n0} - \mathbf{q}_{n1} = 0 \tag{4}$$

are observed on the interface of the media at the points of the singular line

$$\lim (T_0 - T_1) = 0 , \quad \lim (\mathbf{q}_{n0} - \mathbf{q}_{n1}) = 0 , \tag{5}$$

where the limits are taken for $\mathrm{M} \to \mathrm{M}_0$ in the plane normal to the singular line. The body is exposed to external thermal actions. The initial conditions are not specified, since the fact that relations (3) and (4) hold leads to a steady-state regime, as is shown below. Thus, it is necessary to find the solution of Eq. (3) with boundary conditions (4) and (5) and to investigate it.

Temperature Field and Components of the Heat-Flux-Density Vector. We establish the behavior of solutions of the nonstationary heat-conduction equation (2) near the singular points of the conjugation surface *S*.

The homogeneous equation corresponding to (2) is invariant relative to the transformation

$$x = Bx_1 , \quad y = By_1 , \quad z = Bz_1 , \quad t = B^2 t_1 , \quad x = Bx_1 .$$

Allowing for the linearity of conjugation conditions (4) and (5), we establish that the solution of the conjugation problem is determined by the functional equation

$$T_i (x, y, z, t) = A (B)\, T_i (Bx, By, Bz, B^2 t) . \tag{6}$$

Differentiating relation (6) with respect to *B*, we obtain the equation

$$(\mathrm{grad}\, T_i, \mathbf{r}) + 2t \frac{\partial T_i}{\partial t} = KT ,$$

whose solution determines the class of solutions of the heat-conduction equation

$$T_i = x^m \varphi_i \left(\frac{x}{y}, \frac{x}{z}, \frac{x}{\sqrt{t}} \right), \tag{7}$$

which gives, if the representation of the Cartesian coordinates by the curvilinear ones ρ, θ, s according to (1) is allowed for, a representation of the temperature field in the power class for the variable ρ:

$$T = \rho^{m(s)} A\,(\theta, s)\ . \tag{8}$$

Substituting (8) into (3), we obtain the explicit representation of the temperature field

$$T_i = \rho^m\,(C_{1i}\cos\lambda_i\theta + C_{2i}\sin\lambda_i\theta)\ ,\quad \lambda_i = \sqrt{\frac{k_{11}^{(i)}}{k_{22}^{(i)}}}\ . \tag{9}$$

Satisfying conjugation conditions (4), we find a homogeneous system of linear algebraic equations for arbitrary constants. Such a system has a nontrivial solution if its principal determinant is equal to zero. As a result we obtain the singular characteristic equation for determining the index m

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0\ , \tag{10}$$

$$a_{11} = -\,k_{11}^{(1)}\cos m\lambda_1\theta_1 + k_{22}^{(1)}\,\lambda_1\sin m\lambda_1\theta_1\ ,\quad a_{12} = -\,k_{11}^{(1)}\sin m\lambda_1\theta_1 - k_{22}^{(1)}\,\lambda_1\cos m\lambda_1\theta_1\ ,$$

$$a_{13} = k_{11}^{(0)}\cos m\lambda_0\theta_1 - k_{22}^{(0)}\lambda_0\sin m\lambda_0\theta_1\ ,\quad a_{14} = k_{11}^{(0)}\sin m\lambda_0\theta_1 + k_{22}^{(0)}\lambda_0\cos m\lambda_0\theta_1\ ,$$

$$a_{21} = -\,k_{11}^{(1)}\cos m\lambda_1\theta_2 + k_{22}^{(1)}\lambda_1\sin m\lambda_1\theta_2\ ,\quad a_{22} = -\,k_{11}^{(1)}\sin m\lambda_1\theta_2 - k_{22}^{(1)}\lambda_1\cos m\lambda_1\theta_2\ ,$$

$$a_{23} = k_{11}^{(0)}\cos m\lambda_0\theta_2 - k_{22}^{(0)}\lambda_0\sin m\lambda_0\theta_2\ ,\quad a_{24} = k_{11}^{(0)}\sin m\lambda_0\theta_2 + k_{22}^{(0)}\lambda_0\cos m\lambda_0\theta_2\ ,$$

$$a_{31} = \cos m\lambda_1\theta_1\ ,\quad a_{32} = \sin m\lambda_1\theta_1\ ,\quad a_{33} = -\cos m\lambda_0\theta_1\ ,\quad a_{34} = -\sin m\lambda_0\theta_1\ ,$$

$$a_{41} = \cos m\lambda_1\theta_2\ ,\quad a_{42} = \sin m\lambda_1\theta_2\ ,\quad a_{43} = -\cos m\lambda_0\theta_2\ ,\quad a_{44} = -\sin m\lambda_0\theta_2\ .$$

We introduce the heat-flux-density-intensity factors K_1 and K_2 according to the formulas

$$C_{10} = \frac{-\,k_{11}^{(0)}\sin m\lambda_0\theta_1 - k_{22}^{(0)}\lambda_0\cos m\lambda_0\theta_1}{-\,k_{11}^{(0)}\cos m\lambda_0\theta_1 - k_{22}^{(0)}\lambda_0\sin m\lambda_0\theta}\,K_2 + K_1\ ,\quad C_{20} = K_2 - \cot m\lambda_0\theta_1 K_1\ , \tag{11}$$

as a result, the temperature distribution in the matrix will be represented as follows:

$$T_0 = \rho^m\Bigg[(\cos m\lambda_0\theta - \cot m\lambda_0\theta_1\sin m\lambda_0\theta)\,K_1$$

$$+\left(-\frac{k_{11}^{(0)}\sin m\lambda_0\theta_1 + k_{22}^{(0)}\cos m\lambda_0\theta_1}{-\,k_{11}^{(0)}\cos m\lambda_0\theta_1 + k_{22}^{(0)}\sin m\lambda_0\theta_1}\cos m\lambda_0\theta + \sin m\lambda_0\theta\right)K_2\Bigg] + O\,(\rho^{m+1})\ . \tag{12}$$

The heat-flux-density components take the form

$$q_\rho^{(0)} = -k_{11}^{(0)}\rho^{m-1}\left[\left(-\frac{k_{11}^{(0)}\sin m\lambda_0\theta_1 + k_{22}^{(0)}\cos m\lambda_0\theta_1}{-k_{11}^{(0)}\cos m\lambda_0\theta_1 + k_{22}^{(0)}\sin m\lambda_0\theta_1}\cos m\lambda_0\theta + \sin m\lambda_0\theta\right)K_2\right.$$

$$\left. + (\cos m\lambda_0\theta - \cot m\lambda_0\theta_1 \sin m\lambda_0\theta)\, K_1\right] + O(1)\,, \tag{13}$$

$$q_\theta^{(0)} = -k_{22}^{(0)}\rho^{m-1}\left[\left(-\frac{k_{11}^{(0)}\sin m\lambda_0\theta_1 + k_{22}^{(0)}\cos m\lambda_0\theta_1}{-k_{11}^{(0)}\cos m\lambda_0\theta_1 + k_{22}^{(0)}\sin m\lambda_0\theta_1}(-m\lambda_0)\sin m\lambda_0\theta + m\lambda_0\cos m\lambda_0\theta\right)K_2\right.$$

$$\left. + ((-m\lambda_0)\sin m\lambda_0\theta - \cot m\lambda_0\theta_1 m\lambda_0\cos m\lambda_0\theta)\, K_1\right] + O(1)\,,$$

$$q_s^{(0)} = O(1)\,.$$

In the inclusion, we have

$$T_1 = \rho^m\Big\{K_1\,[(d_{11} - \cot m\lambda_0\theta_1 d_{12})\cos m\lambda_1\theta + (d_{21} - \cot m\lambda_0\theta_1 d_{22})\sin m\lambda_1\theta]$$

$$+ K_2\,[(bd_{11} + d_{12})\cos m\lambda_1\theta + (bd_{21} + d_{22})\sin m\lambda_1\theta]\Big\} + O\left(\rho^{m+1}\right),$$

$$q_\rho^{(1)} = -k_{11}^{(1)}m\rho^{m-1}\Big\{K_1\,[(d_{11} - \cot(m\lambda_0\theta_1)\,d_{12})\cos m\lambda_1\theta + (d_{21} - \cot(m\lambda_0\theta_1)\,d_{22})] \tag{14}$$

$$+ K_2\,[(bd_{11} + d_{12})\cos m\lambda_1\theta + (bd_{21} + d_{22})\sin m\lambda_1\theta]\Big\} + O(1)\,,$$

$$q_\theta^{(1)} = -k_{22}^{(1)}m\rho^{m-1}\Big\{K_1\,[-m\lambda_1\,(d_{11} - \cot(m\lambda_0\theta_1)\,d_{12})\sin m\lambda_1\theta + (d_{21} - \cot(m\lambda_0\theta_1)\,d_{22})]$$

$$+ K_2\,[-(bd_{11} + d_{12})\,m\lambda_1\sin m\lambda_1\theta + (bd_{21} + d_{22})\,m\lambda_1\cos m\lambda_1\theta]\Big\} + O(1)\,,$$

$$q_s^{(1)} = O(1)\,,$$

$$d_{11} = \frac{\cos m\lambda_0\theta_2\,(\sin m\lambda_1\theta_2 - \sin m\lambda_1\theta_1)}{\sin m\lambda_1\,(\theta_2 - \theta_1)}\,,\quad d_{12} = \frac{\sin m\lambda_0\theta_2\sin m\lambda_1\theta_2 - \sin m\lambda_1\theta_1\sin m\lambda_0\theta_2}{\sin m\lambda_1\,(\theta_2 - \theta_1)}\,,$$

$$d_{21} = \frac{\cos m\lambda_0\theta_2\cos m\lambda_1\theta_1 - \cos m\lambda_0\theta_1\cos m\lambda_0\theta_1}{\sin m\lambda_1\,(\theta_2 - \theta_1)}\,,\quad d_{22} = \frac{\sin m\lambda_0\theta_2\cos m\lambda_1\theta_1 - \sin m\lambda_0\theta_1\cos m\lambda_1\theta_2}{\sin m\lambda_1\,(\theta_2 - \theta_1)}\,,$$

$$b = \frac{-k_{11}^{(0)}\sin m\lambda_0\theta_1 - k_{22}^{(0)}\lambda_0\cos m\lambda_0\theta_1}{-k_{11}^{(0)}\cos m\lambda_0\theta_1 - k_{22}^{(0)}\lambda_0\sin m\lambda_0\theta}\,.$$

To obtain the particular case of isotropy we set $k_{11}^{(0)} = k_{22}^{(0)} = k_{33}^{(0)} = k_0$; then the characteristic equation (10) gives the known value of the singularity index [2] $m = \pi/(\theta_2 - \theta_1)$.

In the case of the heat-insulated inclusion where we have $k_{11}^{(1)} \to 0$, $k_{22}^{(1)} \to 0$, and $\lambda_1 = \sqrt{k_{22}^{(1)}/k_{11}^{(1)}} \to C = \text{const}$ in (10) we obtain $m = \pi/\lambda_0(\theta_2 - \theta_1)$, and passing to the limit in the homogeneous system of linear algebraic equations, we find that the intensity factor in the distributions (12) and (13) is $K_1 = 0$.

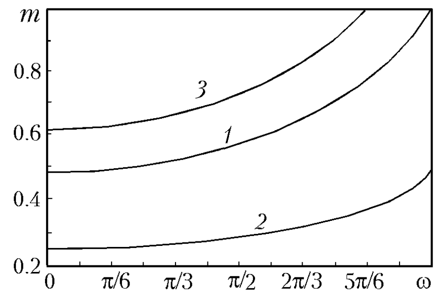

Fig. 2. Order of singularity m for the heat-insulated (isothermal) inclusion vs. angle of opening of the inclusion $\omega = 2\pi - (\theta_2 - \theta_1)$.

For the case of the isothermal inclusion, we obtain $m = \pi/\lambda_0(\theta_2 - \theta_1)$ when $k_{11}^{(1)} \to \infty$, $k_{22}^{(1)} \to \infty$, and $\lambda_1 = \sqrt{k_{22}^{(1)}/k_{11}^{(1)}} \to C_1$ = const in (13), and from the system of linear algebraic equations we find that the intensity factor in the distributions (12) and (13) is $K_2 = 0$.

The change in the order of singularity m for the heat-insulated (isothermal) inclusion as a function of the angle of opening of the inclusion $\omega = 2\pi - (\theta_2 - \theta_1)$ is shown in Fig. 2. Curve 1 corresponds to isotropy, whereas curve 2 corresponds to anisotropy for $\lambda_0 = \sqrt{k_{11}^{(0)}/k_{22}^{(0)}} = 2$, and curve 3 corresponds to $\lambda_0 = \sqrt{k_{11}^{(0)}/k_{22}^{(0)}} = 0.99$.

Discussion of the Results. The distribution of the temperature and the heat-flux-density components (12) and (13) is a superposition of the heat-insulated component characterized by the intensity factor K_2 and the isothermal component with intensity factor K_1.

As follows from (13), the component of heat-flux density in the direction of the axis of the tangent unit vector $\mathbf{n}_1$ has no singularity, which is caused by the redistribution of the heat flux due to the anisotropy of the thermal properties of the heat-conducting material.

Conclusions. For materials with thermal conductivities $k_{11}^{(0)}/k_{22}^{(0)} < 0.25$ in the case of heat-insulated or isothermal boundary surfaces with a singular line, the singularity in the distribution of the heat-flux-density components disappears, which offers a means for optimizing anisotropic inhomogeneous members with uneven interfaces of materials operating under heat loads.

NOTATION

$A(B)$, function of the quantity B; $A(\theta, s)$, functional dependence in the representation of the temperature field; a_{gb} (g, $b = \overline{1, 4}$), matrix elements of the singular characteristic equation; B, arbitrary real number; b, quantity in the distribution of the heat-flux-density components; C_{11}, C_{12}, C_{10}, and C_{20}, arbitrary constants; c_i, specific heat, kcal/(kg·°C); d_{11}, d_{12}, d_{21}, and d_{22}, quantities in the distribution of the heat-flux-density components; h_1, h_2, and h_3, Lamé coefficients; $k_{11}^{(0)}$, $k_{22}^{(0)}$, and $k_{33}^{(0)}$, thermal conductivities of the medium V_0; $k_{11}^{(1)}$, $k_{22}^{(1)}$, and $k_{33}^{(1)}$, thermal conductivities of the medium V_1 (inclusion); k_0, thermal conductivity of the isotropic material; K_1 and K_2, heat-flux-density-intensity factors; K_{0i}, constant; L, smooth singular line which is a set of angular points; m, index of singularity of the heat-flux vector; $\mathbf{n}\mathbf{n}_1\mathbf{n}_2$, moving trihedral of the surface S_0 at the point $M_0 \in S_0$; $\mathbf{n}_1$, tangent vector to the curve L at M_0; $\mathbf{r}_0$ and $\mathbf{r}$, radius vectors of the points M_0 and M respectively; Q_i, quantity of heat released by a unit volume in a unit time; S, surface of anisotropic heat-conducting media; S_1 and S_2, smooth surfaces; S_0, auxiliary surface stretched over the singular line; s, singular-line-arc length reckoned from a certain initial point; $T_i(x, y, z, t)$, temperature; $O(\rho^{m+1})$, quantity of the same order of smallness with the quantity ρ^{m+1}; $O(1)$, quantity equivalent to a constant; $\mathbf{q}_{ni}$, heat-flux-density vector; $q_\rho^{(i)}$, $q_\theta^{(i)}$, and $q_s^{(i)}$, components of the heat-flux-density vector; u, v, orthogonal curvilinear coordinates; V_0 and V_1, regions occupied by the anisotropic body and the inclusion; x, y, z and x_1, y_1, z_1, Cartesian variables; λ_0 and λ_1, ratio of the thermal conductivities of the anisotropic body; ρ, polar radius of the point M; ρ_{0i}, density, kg/m^3; θ_1 and θ_2, values of the curvilinear coordinate θ; φ_i, homogeneous function; ω, angle of opening of the inclusion. Subscripts: ρ, θ, and s determine the components of the heat-flux-density vector in the direction of the corresponding curvilinear coordinate lines; 11, 22, and 33 determine the thermal conductivities for the principal directions.

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